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On the prediction of intermittent turbulent flows

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In this paper we present a theoretical model which permits the conservation equations of fluid dynamics to be conditioned in a fashion analogous to the experimentalist's technique of 'conditioned sampling'. The detailed analysis refers to the best-known sampling condition, outer-edge intermittency; but the model equation may be applicable to other flow situations, wherein conditioning exposes details of the physical phenomena. The analysis results in predictions of the flow variables within the turbulent flow and of the intermittency. Comparison is made with two sets of experimental results for the two-dimensional mixing layer and with a boundary layer.

1. Introduction

This study is motivated by the existing differences between the techniques used daily by the research worker conducting experiments in turbulent shear flows, and by the theoretician developing predictive methods for such flows. The former recognizes from his experiencein the laboratory or in the field that the usual, textbook representation of a turbulent signal as a more-or-less continuous (i.e. unstructured) random variable overlooks essential, physically illuminating features of many turbulent flows. This leads the experimentalist to develop and to employ a variety of so-called conditioned sampling techniques (cf. Kovasznay, Kibens & Blackwelder 1970; Kaplan & Laufer 1968; Coles & Van Atta 1966; Corrsin & Kistler 1955), so that parts, perhaps small parts, of an extended turbulent signal can be statistically analysed without the obfuscation of long time periods devoid of interest. These techniques supplement the usual (unconditioned) time averages with zone averages, point averages, range-conditioned zone averages, etc., and are recognized as providing important information on the physics of turbulent shear flows.

The theoretician developing predictive methods for turbulent shear flows continues to use averaging techniques which overlook developments in the laboratory and in the field.† The well-known method of Reynolds decomposition, and of time-averaging the describing equations of fluid dynamics, corresponds to the unconditioned analysis of the experimentalist. Thus, for example, near the

† This situation has led Kovasznay to observe that the turbulence community consists of experimentalists who do not want to know about predictions and predictors who do not want to know about turbulence.

outer edge of a turbulent shear flow with an interface between the turbulent fluid and the irrotational fluid, the theoretician implicitly overlooks the distinction between the two fluids, and so loses details of the structure in the outer portions of the flow. This conservative, traditional position of the theoretician can be justified if interest is confined to providing the means for estimating quantities such as mean shear, mean heat transfer, etc. For this purpose it is apparently unnecessary to incorporate in a detailed way the existence of the turbulent interface. However, there may be other problems, connected with turbulent mean flows, requiring more detailed behaviour of the outer regions of turbulent flows and more of the physics of the turbulence. The newer, so-called second-order closure methods, which incorporate more of the physics of turbulence into the describing equations, generally provide more accurate descriptions of turbulent shear flows. Likewise, we might expect that analyses which incorporate descriptions of the structure of outer portions of the boundary layer, and more generally descriptions of possibly rare but physically significant events, could be expected to be more accurate and to have wide utility. This expectation may be especially valid for scalars that have constant values in the outer flow.

This situation suggests that the theoretician should have tools of analysis analogous to those used by the experimentalist. Here we consider an approach to an obvious case calling for conditioned equations: that of the interface between turbulent fluid in a shear flow and the irrotational fluid into which it grows. The existence of such an interface is well known from the pioneering work of Corrsin & Kistler (1955); and it provides the earliest application of conditioned statistics in the laboratory. We point out that perhaps the entire range of phenomena in turbulent shear flows, exposed by conditioned sampling in the laboratory, can eventually be described by appropriate extensions of the present point of view. In addition, there may be applications to reacting flows involving oscillating flame sheets. We make a modest start here.

A sensor (e.g. a hot wire giving a signal proportional to the streamwise velocity component) in the outer portions of a turbulent shear flow (e.g. a boundary layer) alternately encounters periods in the irrotational, external flow and periods within the turbulent flow. Kovasznay *et al.* (1970) showed that the velocities within the irrotational fluid are considerably less uniform than was originally believed (cf. Corrsin & Kistler 1955). Nevertheless, that the signal is significantly different in the two parts of the flow is evident qualitatively from observations of an oscilloscope or other device displaying the sensor output. In fact, there has been developed a variety of discrimination techniques permitting a random telegraph signal (i.e. a signal with one of two values, zero or one) to be generated from the output of a variety of sensors: the value zero corresponds to the sensor being in the irrotational fluid, and the value one to it being in the turbulent fluid. With the zero–one signal available, the experimentalist is able to perform a variety of statistical analyses: e.g. to determine the percentage of time the flow is turbulent at the sensor and the mean value, the r.m.s. and other moments of the sensor output associated with the irrotational fluid alone, with the turbulent fluid alone, and with the interfaces, upstream and downstream (i.e. when the zero–one signal changes value).

It is our purpose here to develop an analysis permitting the usual fluid dynamical equations to be conditioned, so as to provide a description of the fluid in the two regions of a turbulent shear flow. We first outline our approach, and develop the general equations describing the mean flow variables within the turbulent fluid alone. We then specialize these equations for turbulent shear flows sufficiently thin that the boundary-layer approximations apply, then compare predictions with experimental data for the two-dimensional mixing layer and for a boundary layer.

2. General analysis

In the course of this analysis we shall need some of the conventional, unconditioned equations of turbulent flows. Thus we start by developing briefly, but in the usual fashion, well-known equations. Consider the equations for a fluid with constant properties in Cartesian co-ordinates and in the usual notation, namely

$$\frac{\partial u_k}{\partial x_k} = 0, \tag{1}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k}(u_i u_k) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k}, \quad i = 1, 2, 3. \tag{2}$$

To be clear, we define the time-average of a quantity $Q(x_1, x_2, x_3, t)$ as

$$\bar{Q}(x_1, x_2, x_3) \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T Q(x_1, x_2, x_3, t)\, dt, \tag{3}$$

and we decompose Q as

$$Q(x_1, x_2, x_3, t) = \bar{Q}'(x_1, x_2, x_3) + Q'(x_1, x_2, x_3, t). \tag{4}$$

Throughout this analysis there will be only one fluctuating component for each fluid dynamical variable (i.e. that indicated by (4)).† Employing these conventional definitions and neglecting viscous transport, we have the following well-known equations, frequently used by theoreticians in the description of turbulent shear flows:

$$\frac{\partial \bar{u}_k}{\partial x_k} = 0, \tag{1a}$$

$$\frac{\partial}{\partial x_k}(\bar{u}_k \bar{u}_i + \overline{u_i' u_k'}) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i}, \quad i = 1, 2, 3, \tag{2a}$$

$$\frac{\partial}{\partial x_k}(\bar{u}_k \overline{u_i' u_j'} + \overline{u_j' u_j' u_k'}) + (\overline{u_i' u_k'})\frac{\partial \bar{u}_j}{\partial x_k} + (\overline{u_j' u_k'})\frac{\partial \bar{u}_i}{\partial x_k}$$
$$= -\left[\frac{\partial}{\partial x_i}\overline{(p' u_j')} + \frac{\partial}{\partial x_j}\overline{(p' u_i')}\right] + \overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} - 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}, \quad i, j = 1, 2, 3. \tag{5}$$

† It will be seen shortly that other kinds of mean quantities appear, and thus that other fluctuating quantities suggest themselves. There appears to be little to be gained from considering them in the present context.

The conditioning function

Suppose we wish to supplement the averaging defined by (3) by one analogous to that which the experimentalist uses to distinguish time intervals when his sensor is in irrotational fluid from those when it is in turbulent fluid. Clearly a suitable zero–one function is needed. To develop an equation for such a function, we adopted the viewpoint of Nee & Kovasznay (1969), Saffman (1970) and others, and postulate a model equation. In our case we can construct a flow situation in which the model equation would describe an identifiable physical variable.† Nevertheless, the viewpoint of a model equation appears the appropriate one. Accordingly, consider a diffusion equation for a scalar quantity $I(x_1, x_2, x_3, t)$,

$$\frac{\partial I}{\partial t} + \frac{\partial}{\partial x_k}(u_k)I) = \dot{w}, \tag{6}$$

with a creation term $\dot{w}$, the volumetric rate of creation of the scalar I with units of $\mathrm{cm}^3/\mathrm{cm}^3\,\mathrm{s}$.

Equation (6) can be used in conjunction with (1) and (2) to develop the well-known equations for the conservation of the mean values, fluxes, intensities, and velocity correlations of a scalar quantity. However, we now imagine that at an arbitrary space-time point I equals either zero or one. This view leads to special correlations of any dynamic variable Q with I. Consider

$$\overline{Q(x_1, x_2, x_3, t)\,(I(x_1, x_2, x_3, t)} = \lim_{T\to\infty} \frac{1}{T}\int_0^T QI\,dt \equiv (\bar{Q})_1 = (\bar{Q}(x_1, x_2, x_3))_1. \tag{7}$$

Clearly by the operation implied by (7), we have accumulated contributions to $\bar{Q}$ only when $I = 1$, and have excluded from the averaging the values of Q when $I = 0$. The resultant mean value is denoted by a subscript one. The only ambiguity in this 'conditioned' mean value of Q is due to 'interfaces' where I changes value. Consistent with the high Reynolds number assumptions used in connexion with (1*a*), (2*a*) and (5), and with the assignment of only zero and unity values to I, these interfaces are considered negligibly small in space; thus their repeated passing of the space point (x_1, x_2, x_3) is assumed to contribute negligibly to the time averaging in (7). In this regard, we are explicit about an assumption taken for granted by the experimentalist when he generates a zero–one signal, then uses it to obtain conditioned statistics.‡

In general $(\bar{Q}')_1$ is not zero although $\bar{Q}' = 0$; and the definition of conditioned means, as in (7), differs by a factor $\bar{I}$ or $(1-\bar{I})$ from that frequently employed by the experimentalist. In our definition, mean values corresponding to those

† Consider a flow involving two fluids. Some of the fluid is distinguished from the rest in a fluid-dynamically insignificant manner (e.g. by its colour). Let $I(x_1, x_2, x_3, t)$ denote the concentration in volumetric percentage of this distinguished or marked fluid. Allow the coloured fluid to increase or decrease by some photochemical process. In this notion (6) is the conservation equation for I with molecular effects neglected. See 'Note added in proof'.

‡ For brevity we shall not always indicate the (x_1, x_2, x_3, t) and (x_1, x_2, x_3) functional dependence of our variables. Throughout we are dealing with one-point, one-time averaging, and with an Eulerian representation.

periods when $I = 0$ (denoted by the subscript zero) are related to unconditioned means, and to means when $I = 1$, by a relation not involving $\bar{I}$, namely

$$\bar{Q} = (\bar{Q})_0 + (\bar{Q})_1. \tag{8}$$

Special cases of (8) are of interest. If $Q \equiv 0$ in the irrotational flow, as is frequently the case with scalar quantities, then $\bar{Q} = (\bar{Q})_1$. If a quantity is statistically the same within the turbulent and irrotational fluids, then $\bar{Q} = (\bar{Q})_1/\bar{I} = (\bar{Q})_0/(1-\bar{I})$. In general, $\lim_{I\to 0} (\bar{Q})_1/\bar{I}$ and $\lim_{I\to 1} (\bar{Q})_0/(1-\bar{I})$ can have any values. Finally, the two-valued, zero–unity, nature of I implies that, if $\bar{I} = 1$, $I \equiv 1$, $\bar{Q} = (\bar{Q})_1$, $\bar{Q}_0 = 0$, and that, if $\bar{I} = 0$, $I \equiv 0$, $\bar{Q} = \bar{Q}_0$, $(\bar{Q})_1 = 0$.

Extension of (7) and inclusion of a second dynamical variable P leads to the following equations, which we shall need subsequently:

$$\left.\begin{aligned} \overline{Q'I} &\equiv (\overline{Q'})_1 = (\bar{Q})_1 - \bar{Q}\bar{I}, \\ \overline{PQI} &\equiv (\overline{PQ})_1 = -\bar{P}\bar{Q}\bar{I} + \bar{P}(\bar{Q})_1 + \bar{Q}(\bar{P})_1 + (\overline{P'Q'})_1. \end{aligned}\right\} \tag{9}$$

Consistent and repeated applications of these rules regarding correlations involving our condition function I respect and assure its zero–unity nature, without the need for explicit specification of $I(1-I) = 0$. We now move on to further statistics following from the availability of I, and thence to the equations from which I and certain other variables of interest can be developed.

If we consider a particular point in space, then the times when the value of I changes form a sequence, $t_1, t_2, \ldots, t_n$, which leads to the point statistics, well known to the experimentalist. If we apply to a variable $Q(x_1, x_2, x_3, t)$ the averaging defined by (3), and consider *all* interface crossings, we have

$$\left.\begin{aligned} \overline{Q\delta(t-t_n)} &= \lim_{T\to\infty} \frac{1}{T}\int_0^T Q\delta(t-t_n)\,dt \\ &= \lim_{N,\,T\to\infty} \frac{N}{T}\left[\frac{1}{N}\sum_{n=1}^{N} Q(x_1, x_2 = X_2, x_3, t_n)\right] \\ &= 2f_I\bar{\bar{Q}}, \end{aligned}\right\} \tag{10}$$

where $X_2(x_1, x_3, t)$ is the location of the interface, f_I is the crossing frequency, and $\bar{\bar{Q}}$ is the ensemble average of Q at interface crossings. The averaging in (10) can be split into a sequence of 'downstream' crossing times when I goes from zero to one and a sequence of an equal number of 'upstream' crossing times when I goes from one to zero†. We show this schematically in figure 1.

Although a slight digression, it seems reasonable at this point to establish an important consequence of (6) and of the averaging discussed here. The mean of (6) becomes an equation for the conditioned velocity components:

$$\frac{\partial}{\partial x_k}(\bar{u}_k)_1 = \bar{\bar{w}} = -\frac{\partial}{\partial x_k}(\bar{u}_k)_0. \tag{11}$$

The conditioned velocity components are not divergence-free.

† Because it seems less ambiguous, we prefer the identification of 'upstream' and 'downstream' crossings to the frequently employed 'backs' and 'fronts', respectively. Upstream crossings correspond to $-\delta(t-t_n) \to \infty$, downstream ones to $\delta(t-t_n) \to \infty$.

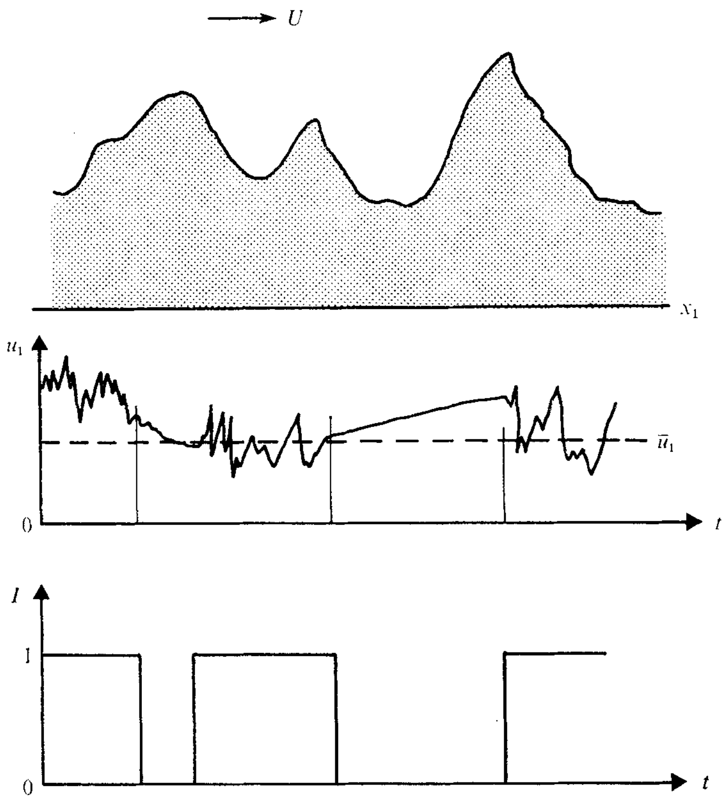

FIGURE 1. Schematic representation of an intermittent flow, and of the u_1 velocity component and its discrimination.

To relate our conditioning function $I(x_1, x_2, x_3, t)$ to the case of outer-edge intermittency requires specification of the nature of $\dot{w}$ and of the distinction between the fluid identified with $I = 1$ and that with $I = 0$. In analyses of other types of conditioning as performed by the experimentalist, other descriptions of $\dot{w}$ and other identifications will be required. Since here the value $I = 1$ will identify turbulent fluid, and since the amount of turbulent fluid increases by an entrainment mechanism associated with the interface between the two fluids, the volumetric rate of creation associated with the $\dot{w}$ term in (6) is due to passages of the interface through the particular space point (x_1, x_2, x_3) being considered (i.e. $\dot{w}$ is non-zero only when I changes value). In terms of a sharp interface, this view suggests that $\dot{w}$ should be thought of as a train of pulses each contributing a positive increment of turbulent fluid, and therefore as a generalized function. Whether entrainment is due to molecular processes at the interface or to engulfment is unimportant in this context. However, this picture of the creation implies that, when $\bar{I} = 0, 1$ (i.e. when there are no interface crossings), $\dot{w} \equiv 0$. This behaviour will be used repeatedly below, when limiting cases corresponding to $\bar{I} = 0, 1$ are considered.

The physical implications of the nature of $\dot{w}$ set out, we postpone the phenomenology of $\dot{w}$, and consider the identification of the conditioning function with turbulence. Although there may be other ways of doing so, our starting point is the equations of Corrsin & Kistler (1955) for the Reynolds stresses in irrotational flow. They show that, in such a flow,

$$\frac{\partial}{\partial x_k}(\overline{u_i' u_k'}) - \frac{1}{2}\frac{\partial}{\partial x_i}(\overline{u_k' u_k'}) = 0, \quad i = 1, 2, 3. \tag{12}$$

In a two-dimensional flow in the x_1, x_2 plane, (12) with $i = 2$ implies that, if the kinetic energy of fluctuations $\overline{u_k' u_k'}$ dies as $x_2^2 \to \infty$, then $\overline{u_1' u_2'}$ approaches a constant value (presumably zero) as $x_2^2 \to \infty$. This is the behaviour we want when $I = 0$. Thus consider

$$(1-I)\, u_k' \left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right) = 0, \quad i = 1, 2, 3, \tag{13}$$

whose first factor is zero in the turbulence, and whose last factor is zero in the irrotational flow. The mean of (13) is

$$\frac{\partial}{\partial x_k}(\bar{u}_k \bar{u}_i \bar{I}) + \frac{\partial}{\partial x_k}[\overline{u_i' u_k'} - (\overline{u_i' u_k'})_1] - \frac{1}{2}\frac{\partial}{\partial x_i}[\overline{u_k' u_k'} - (\overline{u_k' u_k'})_1] - \bar{u}_k \bar{I}\frac{\partial \bar{u}_k}{\partial x_i}$$
$$- (\bar{u}_k)_1 \left(\frac{\partial \bar{u}_i}{\partial x_k} - \frac{\partial \bar{u}_k}{\partial x_i}\right) + \bar{u}_i \overline{u_k' \frac{\partial I}{\partial x_k}} + \overline{u_i' u_k' \frac{\partial I}{\partial x_k}} - \tfrac{1}{2}\overline{u_k' u_k' \frac{\partial I}{\partial x_i}} = \bar{u}_i \bar{w}, \quad i = 1, 2, 3, \tag{14}$$

where we have used (11). In the limiting case of $\bar{I} = 1$, $I \equiv 1$, each term in (14) becomes zero, whereas in the other limiting case $\bar{I} = 0$, $I \equiv 0$, (12) is recovered. For intermediate values of $\bar{I}$ (i.e. for spatial locations involving intermittency, $\bar{I} \neq 0, 1$), (14) provides an apparently new relation among the Reynolds stresses (i.e. $\overline{u_i' u_j'}$, $(\overline{u_i' u_j'})_1$), the turbulent kinetic energies (i.e. $\overline{u_k' u_k'}$, $(\overline{u_k' u_k'})_1$), and a variety of terms, including those of the form $\overline{u_i' u_j' \partial I/\partial x_k}$. In view of the zero–one nature of I, the latter are the point statistics of the interface crossings. In general, these terms should be thought of as examples of the averaging defined by (11), applied separately to the two sequences corresponding to upstream and downstream crossings for which $-\delta(t-t_n) \to \infty$ and $\delta(t-t_n) \to \infty$ respectively. Thus, these terms are zero if the quantity multiplying $\partial I/\partial x_k$ is statistically the same at the two types of crossings. Clearly, to make (14) useful, some modelling is required.

Some conditioned equations

Assuming that we have established an approach to a suitable conditioning function, we may now use it to develop conditioned flow equations. If (2) is multiplied by I, (6) by u_i, addition and averaging in the sense of (3) leads (after some algebra involving use of (4), (7), and (9)) to

$$\frac{\partial}{\partial x_k}[\bar{u}_k((\bar{u}_i)_1 - \bar{u}_i \bar{I})] + \frac{\partial}{\partial x_k}(\overline{u_i' u_k'})_1 + (\bar{u}_k)_1 \frac{\partial \bar{u}_i}{\partial x_k}$$
$$\cong -\frac{1}{\rho}\frac{\partial}{\partial x_i}(\bar{p})_1 + \frac{1}{\rho}\overline{p\frac{\partial I}{\partial x_i}} + \overline{w u_i'}, \quad i = 1, 2, 3, \tag{15}$$

where we employ a high Reynolds number assumption to neglect terms of the form $\nu(\partial^2/\partial x_k \partial x_k)(\bar{u}_i)_1$. We also neglect the terms $\overline{\nu(\partial u_i'/\partial x_k)(\partial I/\partial x_k)}$, under the assumption that the values of $\partial u_i'/\partial x_k$ are insufficiently different on upstream and downstream interfaces to make these terms significant after multiplication by the kinetic viscosity. Equation (15) reduces properly in the two limiting cases, $\bar{I} = 0, 1$.

The procedure used to develop (15) can be extended, e.g. to develop the equations for the conditioned Reynolds stresses $(\overline{u_j' u_i'})_1$, and the conditioned turbulent kinetic energy $(\overline{u_k' u_k'})_1$. The usual procedures of cross-multiplication, addition, averaging and rearranging are followed; but in this case (6) must be included. For present purposes, we shall not require these additional equations.

Boundary-layer flows and modelling

We now specialize the above equations for boundary-layer flows nearly confined to the x_1, x_3 plane, and propose forms for the terms requiring modelling. In making the boundary-layer approximations, we assume that the flow is fully turbulent in the x_1, x_3 plane, and that the conditioned and unconditioned velocity components must therefore be considered of the same order of magnitude. Thus (12) is unaltered, while, if $i = 1$, (14) leads to

$$\frac{\partial}{\partial x_k}(\bar{u}_k u_1 \bar{I}) + \frac{\partial}{\partial x_2}(\overline{u_1' u_2'} - (\overline{u_1' u_2'})_1) - \bar{u}_1 \bar{I}\frac{\partial \bar{u}_1}{\partial x_1} - (\bar{u}_k)_1 \frac{\partial}{\partial x_k}\bar{u}_1$$
$$+ (\bar{u}_1)_1 \frac{\partial \bar{u}_1}{\partial x_1} + \bar{u}_1 \overline{u_k' \frac{\partial I}{\partial x_k}} + \overline{u_1' u_k' \frac{\partial I}{\partial x_k}} - \tfrac{1}{2}\overline{u_k' u_k' \frac{\partial I}{\partial x_1}} \cong \bar{u}_1 \bar{w}. \quad (14a)$$

For $i = 2$, (14) yields

$$\frac{\partial}{\partial x_2}[\overline{u_2' u_2'} - (\overline{u_2' u_2'})_1 - \tfrac{1}{2}(\overline{u_k' u_k'} - (\overline{u_k' u_k'})_1)] - (\bar{u}_1 \bar{I} - (\bar{u}_1)_1)\frac{\partial \bar{u}_1}{\partial x_2}$$
$$+ \overline{u_2' u_k' \frac{\partial I}{\partial x_k}} - \tfrac{1}{2}\overline{u_k' u_k' \frac{\partial I}{\partial x_k}} \cong 0. \quad (14b)$$

If in (14) with $i = 3$ we make the usual assumptions for two-dimensional layers and the assumption of statistical independence of upstream and downstream crossings with respect to $\partial I/\partial x_3$, we obtain no contribution therefrom.

Finally, (15) with $i = 1$ gives

$$\frac{\partial}{\partial x_k}[\bar{u}_k((\bar{u}_1)_1 - \bar{u}_1 \bar{I})] + \frac{\partial}{\partial x_2}(\overline{u_1' u_2'})_1 + (\bar{u}_k)_1 \frac{\partial \bar{u}_1}{\partial x_k} = -\frac{1}{\rho}\frac{\partial}{\partial x_1}(\bar{p})_1 + \frac{1}{\rho}\overline{p\frac{\partial I}{\partial x_1}} + \overline{\dot{w}u_1'}, \quad (15a)$$

whereas $i = 2$ gives

$$0 \cong -\frac{1}{\rho}\frac{\partial}{\partial x_2}\left[\frac{(\bar{p})_1}{\rho} + (\overline{u_2'^2})_1\right] + \frac{1}{\rho}\overline{p\frac{\partial I}{\partial x_2}} + \overline{\dot{w}u_2'}; \quad (15b)$$

and $i = 3$, with some apparently benign assumptions, contributes nothing.

Application of these equations is simplified if we now make the assumptions

$$(\overline{u_1' u_2'}) \cong (\overline{u_1' u_2'})_1, \quad \bar{p} \cong (\bar{p})_1/I, \quad \overline{p\frac{\partial I}{\partial x_1}} \cong \overline{p\frac{\partial I}{\partial x_2}} \cong 0.$$

The first is in accord with experimental data (cf. for example Blackwelder & Kovasznay 1972); it implies that the Reynolds stress $\overline{u_1' u_2'}$ within the irrotational flow is not only constant but zero. The second implies that the mean pressures within the turbulent and irrotational fluids are the same. The third implies that the pressure is statistically identical at upstream and downstream crossings. All three appear reasonable. Their consequence is that we may focus on (11), (14*a*) and (15*a*), and on the modelling necessary to close them, and we may consider that (14*b*) and (15*b*) relate quantities of subsidiary interest.

We follow the usual ideas guiding such modelling: as much physical content as possible, and dimensional consistency. In addition, we anticipate the resulting form of the final equations for the special case of similar flows, and retain a sufficient number of constants associated with the modelled terms that the requisite boundary conditions can be satisfied. Finally, the asymptotic behaviour as $x_2^2 \to \infty$ must be taken into account. While following these ideas, we also seek simplicity, and so recognize that the modelling employed here may be subject to future improvements.

All of the terms to be modelled involve interface crossings and thus must depend on the crossing frequency f_I. We take this dependence to be proportional to $\bar{I}(1-\bar{I})\,\bar{u}_1/\Lambda$, where Λ is a length scale of the order of the boundary-layer thickness†. The creation term in the form $\bar{w}$ is expected to play an important role: Corrsin & Kistler (1955) suggests that the creation of turbulent fluid should depend on the Reynolds stress. Furthermore, dimensional considerations suggest $\bar{w} \propto (|u_1'u_2'|)^{\frac{1}{2}}$. But preliminary results involving comparison with experimental data indicate that the creation of turbulent fluid at the outer edges of a shear layer decays faster than is predicted by the square-root dependence. As we shall see, there are other constraints, related to obtaining proper asymptotic behaviour at such outer edges, so that we are led to

$$\bar{w} = (|u_1'u_2'|/U^2)^{\frac{3}{2}}\,(1-\bar{I})\,\bar{u}_1/\Lambda_1. \tag{16}$$

Λ_1 is a length incorporating a constant into the length scale introduced by the frequency. U is a reference velocity. We have assumed that the dimensionless quantity multiplying the frequency factor is proportional to $\bar{I}^{-1}$, an assumption dictated by asymptotic behaviour.

Next consider the terms involving point statistics at the interfaces. They are difficult to estimate, because it is not clear whether the sequence from the upstream, or that from the downstream, crossings dominates. Here we take them to be proportional to an appropriate power, $\frac{1}{2}$ or 1, of the mean Reynolds stress with the idea that the level of fluctuations of the velocity components is proportional to that stress.‡ Thus we let

$$\overline{u_k'\frac{\partial I}{\partial x_k}} = (|\overline{u_1'u_2'}|/U^2)^{\frac{1}{2}}\,\bar{I}(1-\bar{I})\,\bar{u}_1/\Lambda_2,$$

$$\overline{u_1'u_k'\frac{\partial I}{\partial x_k}} - \tfrac{1}{2}\overline{u_i'u_k'\frac{\partial I}{\partial x_1}} = (\overline{u_1'u_2'}/U^2)\,(\bar{I}(1-\bar{I})\,\bar{u}_1\,U/\Lambda_3. \tag{17}$$

† An examination of the data on f_I suggests that this dependence could as well be $(\bar{I}(1-\bar{I}))^{\frac{1}{2}}$. Consideration of the behaviour as $x_2^2 \to \infty$ suggests the form chosen here. We could alternatively follow Rice (1945), in assuming that the interfacial position is normally distributed. As a consequence, $f_I \propto \partial\bar{I}/\partial x_2$. This alters the nature of the resulting equations so drastically that we prefer the present, more conservative approach.

‡ As will be seen, the choice of the exponents of $\bar{I}$ in (16) and (17) and similar equations is dedicated by considerations of solution behaviour as $x_2^2 \to \infty$.

Λ_2 and Λ_3 are again length scales that are constants times Λ, and that are to be determined.

Finally, we deal with the $\overline{\dot{w}u_1'}$ term. We make it a combination of the creation term and an appropriate power of the mean shear. The $\dot{w}$ factor leads to sampling of the fluctuations of the u_1 velocity component at both upstream and downstream interfaces. These fluctuations are taken to be proportional to $(\overline{u_1'u_2'})^{\frac{1}{2}}$. Thus

$$\overline{\dot{w}u_1'} = (|\overline{u_1'u_2'})/|U^2)^{\frac{5}{4}}(1-\bar{I})\,\bar{u}_1 U/\Lambda_4. \tag{18}$$

Λ_4 is the final length scale introduced by the modelling.

With the above assumptions and modelling, the final equations of intermittent turbulence we deal with here are

$$\left.\begin{aligned}
&\frac{\partial}{\partial x_k}(\bar{u}_k)_1 = (|\overline{u_1'u_2'})/|U^2)^{\frac{3}{4}}(1-\bar{I})\,\bar{u}_1/\Lambda_1,\\
&\frac{\partial}{\partial x_k}(\bar{u}_k\bar{u}_1\bar{I}) - \bar{u}_1\bar{I}\frac{\partial \bar{u}_1}{\partial x_1} - (\bar{u}_k)_1\frac{\partial \bar{u}_1}{\partial x_k} + (\bar{u}_1)_1\frac{\partial \bar{u}_1}{\partial x_1} + [\bar{u}_1(|\overline{u_1'u_2'}|U^2)^{\frac{1}{2}}\bar{I}(1-\bar{I})/\Lambda_2\\
&\quad + (\overline{u_1'u_2'}/U^2)\,\bar{I}(1-\bar{I})\,U/\Lambda_3]\,\bar{u}_1 = \bar{u}_1^2(|\overline{u_1'u_2'}|/U^2)^{\frac{3}{4}}(1-\bar{I})/\Lambda_1,\\
&\frac{\partial}{\partial x_k}[\bar{u}_k((\bar{u}_1)_1 - \bar{u}_1\bar{I})] + (\bar{u}_k)_1\frac{\partial \bar{u}_1}{\partial x_k} + \frac{\partial}{\partial x_2}\overline{u_1'u_2'}\\
&\quad = -\frac{\bar{I}}{\rho}\frac{\partial \bar{p}}{\partial x_1} + (\overline{u_1'u_2'}/U^2)^{\frac{5}{4}}(1-\bar{I})\,\bar{u}_1 U/\Lambda_4.
\end{aligned}\right\} \tag{19}$$

If in these equations $(\bar{u}_k)_1 = 0$ when $\bar{I} = 0$, they are identically satisfied term by term. If $(\bar{u}_k)_1 = \bar{u}_k$ when $\bar{I} = 1$, then the second equation is satisfied identically, and the first and third become the usual equations for unconditioned turbulent flows of the boundary-layer type.

For a complete formulation of an intermittent flow in the present context, (19) must be supplemented with equations for the unconditioned velocity components $\bar{u}_1$ and $\bar{u}_2$, and for the mean shear stress $\overline{u_1'u_2'}$ (i.e. (1a) and (2a) with $i = 1$ and boundary-layer approximations invoked). The nature of the resulting set of equations will be determined by the closure scheme employed for these supplemental equations. It seems appropriate for present purposes to take the view that the unconditioned flow is known, and that the conditioned variables are to be determined from (19). For this purpose (19) can be written formally as

$$\frac{\partial}{\partial x_k}(\bar{u}_k)_1 = \mathscr{R}_1, \quad \frac{\partial}{\partial x_k}(\bar{u}_k\bar{I}) = \mathscr{R}_2, \quad \frac{\partial}{\partial x_k}(\bar{u}_k(\bar{u}_1)_1) = \mathscr{R}_3.$$

$\mathscr{R}_i$, $i = 1, 2, 3$ are functions of the known quantities, $\bar{u}_1$, $\bar{u}_2$, $\bar{p}$, $u_1'u_2'$, and of the dependent quantities, $(u_1)_1$, $(\bar{u}_2)_1$, $\bar{I}$ in finite form. Standard methods show that these are hyperbolic equations with the streamlines as double characteristics and lines $x_1 =$ constant as single characteristic lines. Along the streamlines, it is clear from the above that

$$\frac{d\bar{I}}{d\xi} = \mathscr{R}_2, \quad \frac{d(u_1)_1}{d\xi} \equiv \mathscr{R}_3.$$

Consider the appropriate initial and boundary data. We assume for the moment that along the x_1 axis the flow is fully turbulent. Since the unconditioned flow is taken as given, and the flow is taken to be fully turbulent along the x_1 axis, $(\bar{u}_1)_1(x_1, 0)$ and $(\bar{u}_2)_1(x_1, 0)$ are known and $\bar{I}(x_1, 0) = 1$. At an initial station (e.g. $x_1 = 0$), $(\bar{u}_1)_1(0, x_2)$ and $\bar{I}(0, x_2)$ may be specified. On physical grounds, these initial distributions must have an appropriate behaviour relative to the known unconditioned flow as x_2 increases: namely, $(\bar{u}_1)_1(0, x_2) < \bar{u}_1(0, x_2)$, $0 < \bar{I}(0, x_2) < 1$ and $\lim_{x_2 \to \infty} (\bar{u}_1)_1(0, x_2), \bar{I}(0, x_2) = 0$ (i.e. the initial data must correspond to a turbulent flow bounded by an irrotational flow on one side, $x_2 > 0$). If a streamline passes through a point $x_1 = 0$, x_2 sufficiently large that $(\bar{u}_1)_1 \simeq \bar{I} \simeq 0$, then the above $\mathscr{R}_2$ and $\mathscr{R}_3$ functions along that streamline are zero as long as $\partial \bar{u}_1/\partial x_2 \simeq 0$, indicating that $(\bar{u}_1)_1$ and $\bar{I}$ will remain zero until that streamline 'enters' the boundary layer. Thus specification of boundary data along $x_2 = 0$ and of appropriate initial data at $x_1 = 0$ assures proper behaviour of $(\bar{u}_1)_1$ and $\bar{I}$ at $x_2 = 0$, $x_2 \to \infty$ for arbitrary $x_1 > 0$.

The situation regarding $(\bar{u}_2)_1$ is somewhat different. No initial data with respect to this conditioned velocity component can be specified. Rather $(\bar{u}_2)_1\ (0, x_2)$ is obtained from the initial data on $(\bar{u}_1)_1$ and $\bar{I}$. The requirement that

$$\lim_{x_2 \to \infty} (\bar{u}_2)_1 (x_1, x_2) = 0$$

is considered to impose a constraint on the lengths scales Λ_i, $i = 1, \ldots, 4$ (e.g. one of them is related to the others so that this condition is satisfied).

3. Application to the two-dimensional mixing layer

We now apply the equations developed in §2 to a simple turbulent flow. We seek a flow which is described by a similarity solution (so that the numerical analysis is reduced in complexity), and which has been studied experimentally (so that its intermittency and conditioned velocity components are available). Such an initial application has the further advantage of providing a means of estimating in a largely formal way the length scales Λ_i, $i = 1, \ldots 4$, appearing in the modelling. Whether these same length scales will apply to more general cases of turbulent intermittent flows remains to be determined.

The two-dimensional, free-mixing layer is similar in terms of the variable $\eta = x_2/x_1$. Moreover, there are available data of Wygnanski & Fiedler (1970) and of Spencer & Jones (1971) on the cases of mixing with one stream quiescent, and with both moving, respectively. We thus consider application of our analysis to two-dimensional free-mixing for which the pressure is constant. The flow is shown schematically in figure 2.

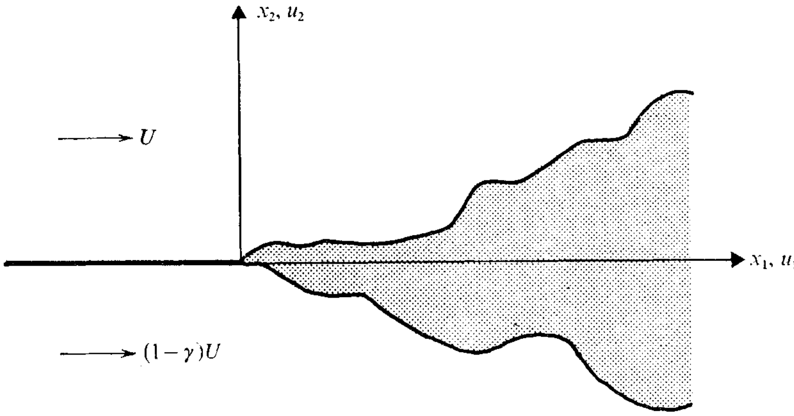

FIGURE 2. Schematic representation of a two-dimensional mixing layer.

Unconditioned flow

Following the philosophy employed above, we assume that the unconditioned velocity components and mean shear stress are known, and we compute the two conditioned velocity components and the intermittency. The unconditioned flow is described in terms of a stream function $f(\eta)$ according to

$$ff'' = T'. \tag{20}$$

Primes denote differentiation with respect to η. The velocity components are related to f by

$$\bar{u}_1 = Uf', \quad \bar{u}_2 = -(f - \eta f').$$

Also, $T = \overline{u_1' u_2'}/U^2$, a non-dimensional Reynolds stress.

Instead of assuming either a model for eddy viscosity or another closure scheme, we follow a more direct approach, suitable for present purposes, and assume a form for $f''(\eta)$. Then quadrature gives f, f' and T. In carrying out these quadratures, we impose the conditions $f(0) = 0$ (i.e. we place the origin of x_2 at the dividing streamline) and $T(\pm\infty) = 0$. In particular, we assume

$$f'' = \gamma\sigma/\pi^{\frac{1}{2}} \exp[-\sigma^2(\eta+\eta_0)^2], \tag{21}$$

which involves the two parameters σ (the spreading parameter, which is given) and η_0 (which must be determined from the conditions on $T(\pm\infty)$). Quadrature yields

$$\left.\begin{aligned} f' &= (1-\gamma) + \frac{\gamma}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\sigma(\eta+\eta_0)} \exp[-\xi^2]\,d\xi = (1-\tfrac{1}{2}\gamma) + \tfrac{1}{2}\gamma \operatorname{erf}(\sigma(\eta+\eta_0)), \\ f &= (1-\gamma)\eta + \frac{\gamma}{\pi^{\frac{1}{2}}}\int_0^{\eta} d\xi' \int_{-\infty}^{\sigma(\xi'+\eta_0)} \exp[-\xi^2]\,d\xi = (1-\tfrac{1}{2}\gamma)\eta + \frac{\gamma}{2\sigma}\int_{\sigma\eta_0}^{\sigma(\eta+\eta_0)} \operatorname{erf}\xi\,d\xi; \end{aligned}\right\} \tag{22}$$

and (20) gives

$$T = -\int_{\eta}^{\infty} f'(1-f')\,d\eta - f(1-f'). \tag{23}$$

If $T(-\infty) = 0$, (23) requires that η_0 be selected so that

$$0 = \int_0^{\infty} f'(1-f')\,d\eta + \int_{-\infty}^{0} f'(1-\gamma-f')\,d\eta.$$

After some calculation, this leads to

$$[(2-\gamma)/\gamma]\,\sigma\eta_0 + \int_0^{\sigma\eta_0} \operatorname{erf}\xi\,d\xi = \int_0^{\infty} (\operatorname{erf}\xi - \operatorname{erf}^2\xi)\,d\xi, \tag{24}$$

which determines $\sigma\eta_0$.

If we were able to compare predicted and measured characteristics of the u_2 velocity components, we should have to re-interpret our solutions in terms of a co-ordinate system whose origin is not on the dividing streamline $f = 0$. Comparison of the x_1 velocity component involves only a translation of η.

Equations for conditioned velocities and intermittency

Introduce two new non-dimensional variables in (19): namely,

$$\tilde{u}_1 \equiv (\bar{u}_1)_1/U \quad \text{and} \quad \tilde{u}_2 \equiv (\bar{u}_2)_1/U.$$

Then (19) in similarity form leads to

$$\left.\begin{aligned} \tilde{u}_2' &= \eta\tilde{u}_1' + \kappa f'(|T|)^{\frac{3}{4}}(1-\bar{I}), \\ \bar{I}' &= -\frac{(f'(\kappa|T|^{\frac{3}{4}} - \delta\bar{I}|T|^{\frac{1}{2}}) - \beta T\bar{I})(1-\bar{I})}{f} - \frac{f''(-(f-\eta f')\bar{I} - \tilde{u}_2)}{ff'}, \\ \tilde{u}_1' &= f'' + \frac{f''\eta}{f}(f'\bar{I} - \tilde{u}_1) - \frac{f'(f'(\kappa|T|^{\frac{3}{4}} - \delta\bar{I}|T|^{\frac{1}{2}}) - (\beta T\bar{I} + \phi|T|^{\frac{5}{4}})(1-\bar{I})}{f}. \end{aligned}\right\} \tag{25}$$

The four length scales Λ_i, $i = 1, \ldots 4$, are replaced by the constants β, δ, κ, ϕ according to $\kappa\Lambda_1 = \delta\Lambda_2 = \beta\Lambda_3 = \phi\Lambda_4 = x_1$ (i.e. similarity requirements necessitate that the length scales increase linearly with the x_1 co-ordinate). Negative values of β, δ, κ and ϕ are not prohibited, but would simply imply that appropriate negative signs should have been introduced in the modelling (16)–(18). On physical grounds, we expect $\kappa > 0$, since it relates to the increase in turbulent fluid at the interfaces; but the remaining constants arise from point statistics at the interfaces, and are therefore ambiguous, even as to sign.

Three-point boundary conditions are to be applied to (25). At $\eta = 0$, we take the flow to be fully turbulent, so that

$$\tilde{u}_1(0) = f'(0), \quad \tilde{u}_2(0) = 0, \quad \bar{I}(0) = 1. \tag{26}$$

At $\eta \to \pm\infty$ the flow is fully irrotational, so that

$$\tilde{u}_1(\pm\infty) = \tilde{u}_2(\pm\infty) = \bar{I}(\pm\infty) = 0. \tag{27}$$

In this formulation, we do not allow for intermittency at the dividing streamline. Experimental data show that $0{\cdot}95 \lesssim \bar{I}(0) \lesssim 0{\cdot}99$; so for present purposes we feel justified in making this assumption. However, an alternative formulation would involve specifying only (27), and altering the modelling and/or the strategy of solution, so that the solutions of (25) would go smoothly through the origin, where $f = 0$.† Here we effectively divide the mixing layer into two separate intermittent flows, one for $\eta > 0$, the other for $\eta < 0$, and permit separate length scales (i.e. separate constants β, δ, κ and ϕ for each). We rationalize this by noting that the two interface surfaces are different on the two sides of the mixing layer, so that the entrainment mechanism and the various point statistics are expected to be different on the two sides.

This point of view also has the advantage that the resulting constants may be interpreted as loosely applying to special boundary-layer flows growing linearly with x_1, with a constant slip velocity at the wall. For $\eta > 0$ the velocity profiles would be conventional, in that the slip velocity is less than U (i.e. $0 < f'(0) < 1$). For $\eta < 0$ the solutions would have to be interpreted as applying to the case of a boundary layer over a moving wall.

If the boundary conditions at $\eta \to \pm\infty$ are dropped, a solution of (25) for any values of β, δ, κ and ϕ is $\bar{I} \equiv 1$, $\tilde{u}_1 \equiv f'$, $\tilde{u}_2 \equiv -(f-\eta f')$ (i.e. the conditioned and unconditioned flow variables are identical if the flow is fully turbulent). This implies that special attention must be devoted to the behaviour near the origin for solutions that do satisfy the boundary conditions at $\eta \to \pm\infty$. In fact, the behaviour at the origin and as $\eta^2 \to \infty$ is important in deciding on the strategy to be followed in finding the solutions. Accordingly, we consider these two limiting situations. Suppose that, near $\eta = 0$, we assume

$$\left.\begin{aligned} &f \simeq f'(0)\,\eta + \tfrac{1}{2}f''(0)\,\eta^2 + \ldots, \quad \tilde{u}_1 \simeq f'(0) + f''(0)\,\eta + \tfrac{1}{2}\tilde{u}_1''(0)\,\eta^2 + \ldots, \\ &\tilde{u}_2 \simeq \tfrac{1}{2}f''(0)\,\eta^2 + \ldots, \quad \bar{I} \simeq 1 + \tfrac{1}{2}\bar{I}''(0)\,\eta^2 + \ldots. \end{aligned}\right\} \tag{28}$$

Then substitution into (25) leads to the following results. If β, δ and κ are constrained so that

$$1 = \frac{f'(0)\,(\kappa|T(0)|^{\frac{3}{4}} - \delta|T(0)|^{\frac{1}{2}}) - \beta T(0)}{2f'(0)}, \tag{29}$$

then $\bar{I}''(0)$ may be arbitrarily specified, and

$$\tilde{u}_1''(0) = (f'(0) + \tfrac{1}{2}\phi|T(0)|^{\frac{5}{4}})\,\bar{I}''(0). \tag{30}$$

Imposition of $\bar{I}''(0) < 0$ forces the solutions away from the uninteresting ones corresponding to the unconditioned flow.

Consider next the behaviour at $\eta \to \infty$. The unconditioned flow becomes

$$f \simeq \eta - \alpha, \quad f' \simeq 1.$$

Thus from (23) provided $\sigma(\eta+\eta_0) \gg 1$, we obtain the approximation

$$-T \simeq \frac{\gamma}{2\pi^{\frac{1}{2}}\sigma}\exp\left[-\sigma^2(\eta+\eta_0)^2\right]\left[1 - \frac{\eta_0+\alpha}{\eta+\eta_0} + O\left(\frac{1}{\sigma^2(\eta+\eta_0)^2}\right)\right]. \tag{31}$$

Now, in (25) as $\eta \to \infty$, the $|T|^{\frac{3}{4}}$ terms dominate, and we obtain

$$\tilde{u}_1' \simeq -\frac{\kappa|T|^{\frac{3}{4}}}{\eta-\alpha}, \quad \tilde{u}_2' \simeq \eta\tilde{u}_1' + \kappa|T|^{\frac{3}{4}}, \quad \bar{I}' \simeq -\frac{\kappa|T|^{\frac{3}{4}}}{\eta-\alpha}. \tag{32}$$

† As we shall see, the present equations involve $(1-\bar{I})/f \propto \eta$ as $\eta \to 0$. If $\bar{I}(0) \neq 1$, a constraint on the constants β, δ, κ and ϕ, similar to (29), will prevail.

With (31) employed in (32), we obtain analytic solutions satisfying the boundary conditions at $\eta \to \infty$, subject to the inequality on $\sigma(\eta+\eta_0)$; namely,

$$\tilde{u}_1 \simeq \bar{I}, \quad \tilde{u}_2 \simeq \alpha\bar{I}, \quad \bar{I} \simeq \kappa\left(\frac{\gamma}{2\pi^{\frac{1}{2}}\sigma}\right)^{\frac{1}{2}} \frac{\exp\left[-\frac{1}{2}\sigma^2(\eta+\eta_0)^2\right]}{\sigma^2(\eta-\alpha)(\eta+\eta_0)}. \tag{33}$$

Several remarks about (33) are indicated. If κ is specified, then (33) provide the starting values for an inward integration at an appropriately large value of η. The quotients $(\tilde{u}_1/\bar{I})$ and $(\tilde{u}_2/\bar{I})$ represent the mean values of the conditioned velocities within the turbulence alone. We see that, as $\eta \to \infty$, these quotients approach unity and α, respectively, indicating that the turbulent fluid takes on the velocity of the external fluid. As suggested earlier, our modelling has been guided by the apparent desirability of this behaviour. It is reasonably easy to construct models that lead to $(\tilde{u}_I/I) = 0, \infty$, and that we consider unacceptable. Making $\bar{w} \propto (\overline{u_1'u_2'}/\bar{I})^{\frac{1}{2}}$ is such a model.

The experimental data on the asymptotic behaviour of $(\tilde{u}_1/\bar{I})$ are somewhat ambiguous, indicating a value of unity or a value somewhat less than unity. The corresponding data on the behaviour of $(\tilde{u}_2/\bar{I})$ are scarce. But Kovasznay *et al.* (1970) show that $(\tilde{u}_2/\bar{I})-(\bar{u}_2/U)$ does not approach zero at the outer edge of a turbulent boundary layer. We are unable to adjust our modelling to achieve such behaviour. We therefore proceed with the physically reasonable behaviour given by (31).

For $\eta \to -\infty$ a similar analysis applies, provided $\gamma > 1$. The unconditioned flow behaves as

$$f \simeq (1-\gamma)\eta-\alpha, \quad f' \simeq 1-\gamma.$$

Equation (31) is modified but we find

$$\tilde{u}_1 = (1-\gamma)\bar{I}, \quad \tilde{u}_2 = \alpha\bar{I}.$$

$\bar{I}$ is given by (33) multiplied by $(1-\gamma)$. The two α quantities for $\eta \to \pm\infty$ are different and obtained from the unconditioned flow. The case $\gamma = 1$ requires special treatment, which does not appear warranted. We thus confine ourselves to describing the high-speed side of layers mixing with quiescent fluid, and layers with $\gamma > 1$.

On the basis of the end-point analyses given here, a strategy for obtaining solutions is indicated. Consider the situation for $\eta > 0$. We specify $\bar{I}''(0)$ so as to achieve agreement between prediction and experimental data, and perform two integrations. We carry out an inward integration from a suitably large value of η, so that the asymptotic solutions given by (33) apply. We carry out an outward integration, starting from the origin, employing (28)–(31). At an intermediate value of η, the values of the three, unconstrained constants among β, δ, κ and ϕ are selected so that continuity in $\tilde{u}_1$, $\tilde{u}_2$ and $\bar{I}$ is achieved. A similar technique applies to the other half-plane $\eta < 0$.†

Consider first the high-speed side of a mixing layer with quiescent fluid on the low-speed side: $\gamma = 1$. To facilitate comparison with the data for this flow from

† The actual numerical effort required to exploit this strategy turns out to be non-trivial, and several methods failed to converge. However, quasi-linearization, based on treating β, κ and ϕ as parameters with the related δ of (29), is found to be reasonably efficient.

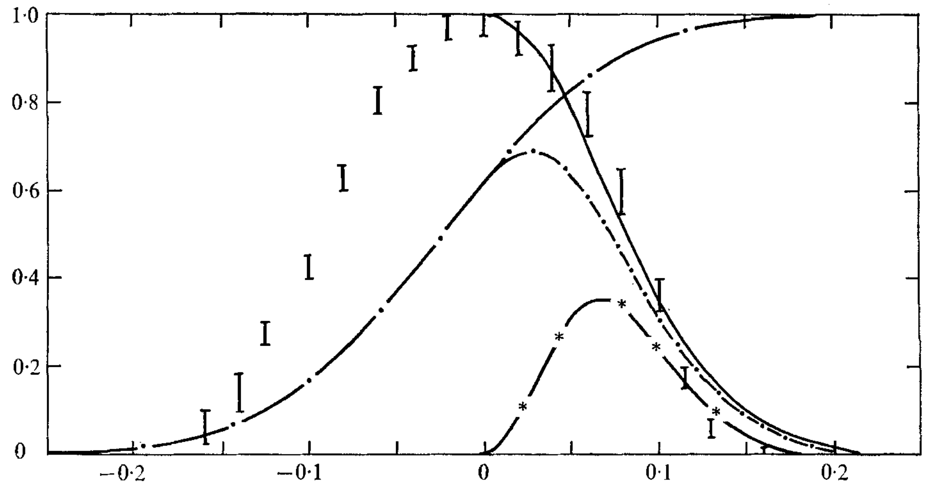

FIGURE 3. The distributions of the velocity components and of intermittency in a free-mixing layer, $\gamma \simeq 1$: —·—, $\overline{u}_1/U$; ——, $\bar{I}$; –·–·–·–, $\tilde{u}_1$; —*—, $\tilde{u}_2 \times 10^2$; I, experimental data on $\bar{I}$ from Wygnanski & Fiedler (1970).

σ	γ		β	δ	κ	ϕ	$\bar{I}''(0)$	η_e
9·0	1·0	$\eta > 0$	−7·878	−2·295	41·78	53·76	−150	0·16
20·4	0·7	$\eta > 0$	−118	−8·639	106·4	163·1	−500	0·070
—	—	$\eta > 0$	619	86·05	144·8	−320·6	−500	—
77·0	—	$\eta > 0$	−1197	−29·18	711·5	1604	—	0·017

TABLE 1. Parameters identifying the unconditioned flows and the solution parameters for the conditioned flow variables

Wygnanski & Fiedler (1970), we take $\sigma = 9$. Figure 3 shows the predicted distributions of $\overline{u}_1$, $\overline{u}_2$ and $\bar{I}$ along with the experimental data for $\bar{I}$ on both sides of the flow. The values of the parameters obtained for this case are given in table 1. Agreement between prediction and experiment is reasonably good in the inner portions of the flow where the intermittency is high; but the predicted decay to the external flow is slower than indicated by the data. This behaviour will be found in all of the cases considered here, suggesting some changes in the modelling.

We can make additional comparisons of prediction and experiment. Consider the conditioned streamwise velocity component. In general the differences among the three values of the u_1 velocity component, $\overline{u}_1$, $(\overline{u}_1)_1$ and $(\overline{u}_1)_0$, are so small that the slight differences between the assumed and experimental unconditioned velocity profile are significant for this comparison. Accordingly, we present in figure 4 the differences between these components. The comparison is seen to be reasonably good, again except near the outer edge, where the previously mentioned slow decay of the predicted intermittency leads to a slower decay of the velocity difference between the turbulent and the external fluid.

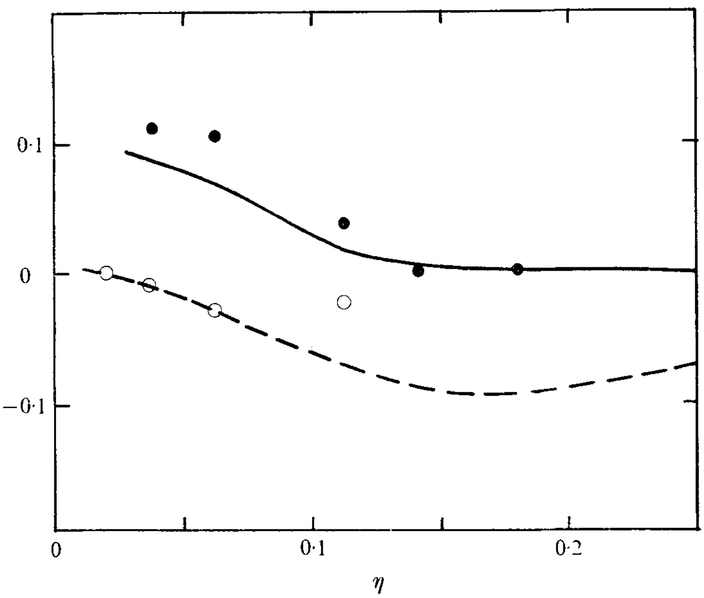

FIGURE 4. Distribution of zone averages of streamwise velocity component in a free-mixing layer, $\gamma = 1$: ——, $((\bar{u}_1/U) - \tilde{u}_1)/(1-\bar{I}) - (\bar{u}_1/U)$; – – –, $(\tilde{u}_1/\bar{I}) - (\bar{u}_1/U)$; ●, ○, experimental data on $\bar{I}$ from Wygnanski & Fiedler (1970).

We cannot make an absolute prediction of the crossing frequency, since it appears only as part of the various modelled terms. But we can compute the distribution of the crossing frequency normalized with respect to its maximum value, and compare this distribution with data, as shown in figure 5. It will be seen that, because the intermittency is between 0·95 and unity on the axis in the experiments of Wygnanski & Fiedler, the computed distribution is in error near the axis. Near the outer edge good agreement is obtained.

Similar results are found for the mixing layer studied by Spencer & Jones (1971) for $\gamma = 0{\cdot}7$, $\sigma = 20{\cdot}4$. Figure 6 gives the predicted distributions of $\tilde{u}_1, \tilde{u}_2$ and $\bar{I}$, on both sides of the layer, along with the intermittency data. The values of the parameters obtained from the analysis are also given in table 1. Again the agreement between prediction and experiment with respect to the intermittency is quite good in the middle part of the flow, but the predicted approach to the external flows is slower than the data indicate.

Spencer & Jones did not give any data on crossing frequency; but we are able in figure 7 to compare our results in terms of the several u_1 velocity components. Again to remove the small differences between the assumed and measured unconditioned velocities, we present the comparison in terms of differences. Except for the more rapid approach of the data to the free-stream values near the outer edges of the layer, the agreement is satisfactory.

At this stage in the development of the analysis, there appears little point in comparison with additional data on the mixing layer. We have in fact performed the numerical analysis of the second case of Spencer & Jones, $\sigma = 50{\cdot}4$, $\gamma = 0{\cdot}4$, with results such as are shown here.

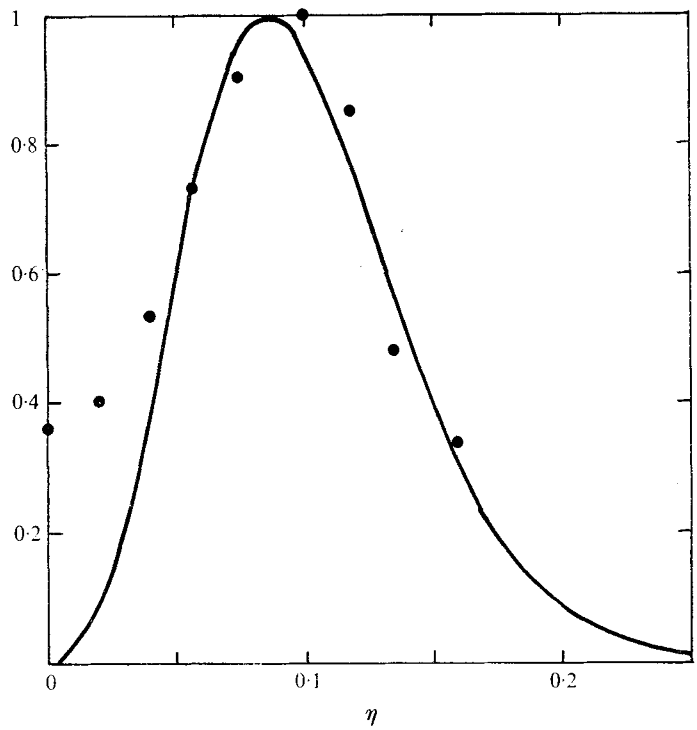

FIGURE 5. Distribution of normalized crossing frequency $f_I/(f_I)_{max}$ on the high velocity side of a free-mixing layer, $\gamma \simeq 1$: ——, prediction; ●, experimental data from Wygnanski & Fiedler (1970).

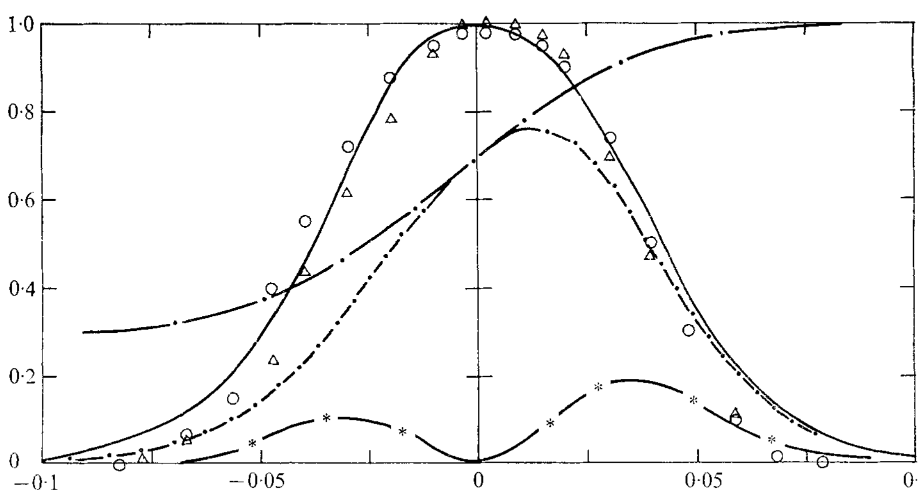

FIGURE 6. The distributions of the velocity components and of intermittency in a free-mixing layer, $\gamma = 0{\cdot}7$: —·—, $\bar{u}_1/U$; ——, $\bar{I}$; -·-·-·-, $\bar{u}_1$; —*—, $\tilde{u}_2 \times 10^2$; △, ○, experimental data on $\bar{I}$ from Spencer & Jones (1971).

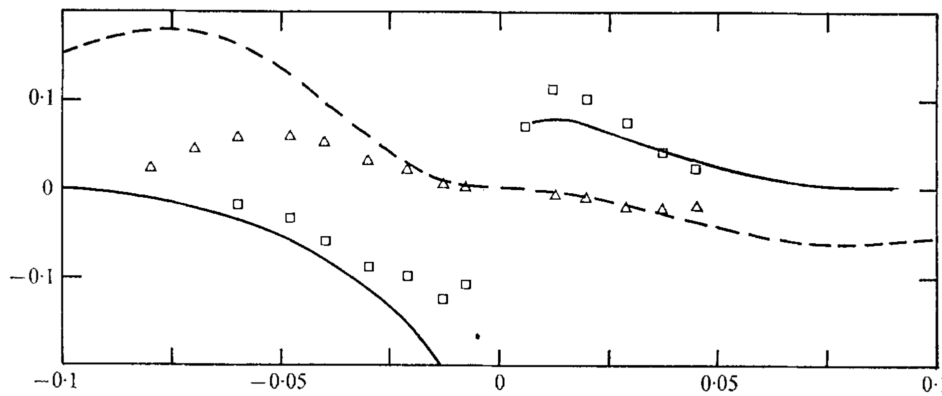

FIGURE 7. The distributions of zone averages of the streamwise velocity components in a free-mixing layer, $\gamma = 0{\cdot}7$: ——, $((\bar{u}_1/U) - \tilde{u}_1)/(1-\bar{I}) - (\bar{u}_1/U)$; ----, $(\tilde{u}_1/\bar{I}) - (\bar{u}_1/U)$.

4. Application to a simulated boundary layer

The analysis for the mixing layer does not strictly apply to a boundary layer with its nonlinear growth. However, it is possible to adjust the parameters appearing in the assumed unconditioned flow so as to simulate a boundary layer, in order to permit comparison with the measurements of Kovasznay *et al.* (1970) without redoing the entire analysis. Accordingly, we select σ, γ and η_0 in (21) so that our unconditioned flow reasonably approximates the flow conditions of their experiment. In particular, we match their shearing velocity at the wall and the slip velocity at the wall ($T(0) = -0{\cdot}00207, f'(0) = 0{\cdot}4$). In addition, we adjust the growth of the layer to approximate that which existed locally in their experiment. We find that $\sigma = 77$, $\gamma = 3$, $\eta_0 = 0{\cdot}0077$.

To achieve agreement for the intermittency, we also modify slightly the strategy indicated above for the mixing layers, as follows. We set $\bar{I}''(0) = 0$, and adjust the intermediate point at which the two solutions obtained by inward and outward integrations are made continuous. In this case, the second solution is simply given by the unconditioned flow with $\bar{I} = 1$.

The results of this computation are shown in figure 8, along with the various sets of data for intermittency given by Kovasznay *et al.*, depending on the particular experiment performed.† The values of the parameters we obtain are given in table 1. We present our results in terms of x_2/δ, where δ is defined by the value of x_2 corresponding to $f' = 0{\cdot}99$, the definition used by Kovasznay *et al.* Again it will be seen that the agreement over the inner portions of the flow is reasonably good, but that the approach to the external stream is somewhat slower than the data would suggest.

In this case, the agreement between our assumed unconditioned velocity profile and the experimental one is sufficiently good that we can make the comparison of the several streamwise velocities directly, as shown in figure 9. Generally, the agreement is quite satisfactory. We have attempted a similar comparison for the

† They use analog techniques.

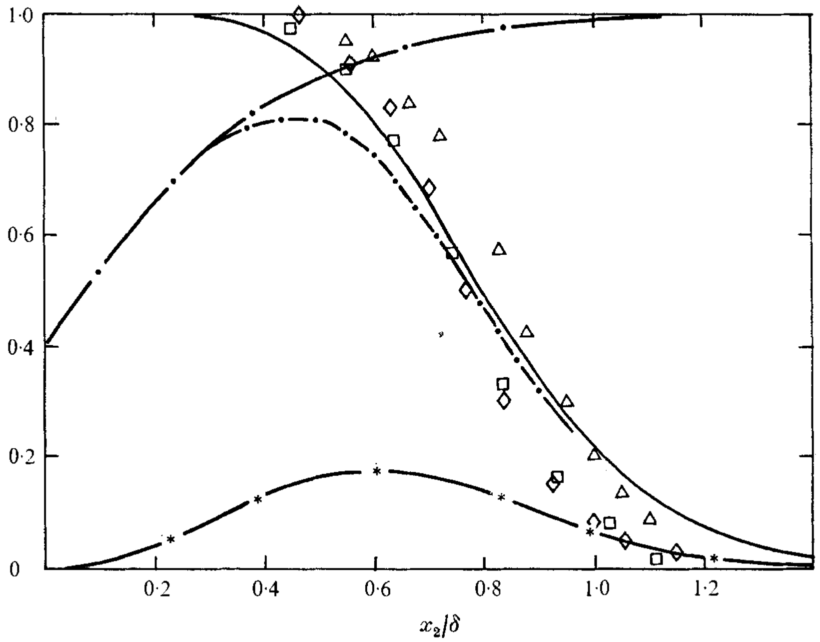

FIGURE 8. The distributions of the velocity components and of the intermittency in a simulated boundary layer: —·—, $\bar{u}_1/U$; ——, $\bar{I}$; —·—·—, $\tilde{u}_1$; —*—, $\tilde{u}_2 \times 10^2$; □, ◇, △, experimental data on $\bar{I}$ from Kovasznay *et al.* (1970).

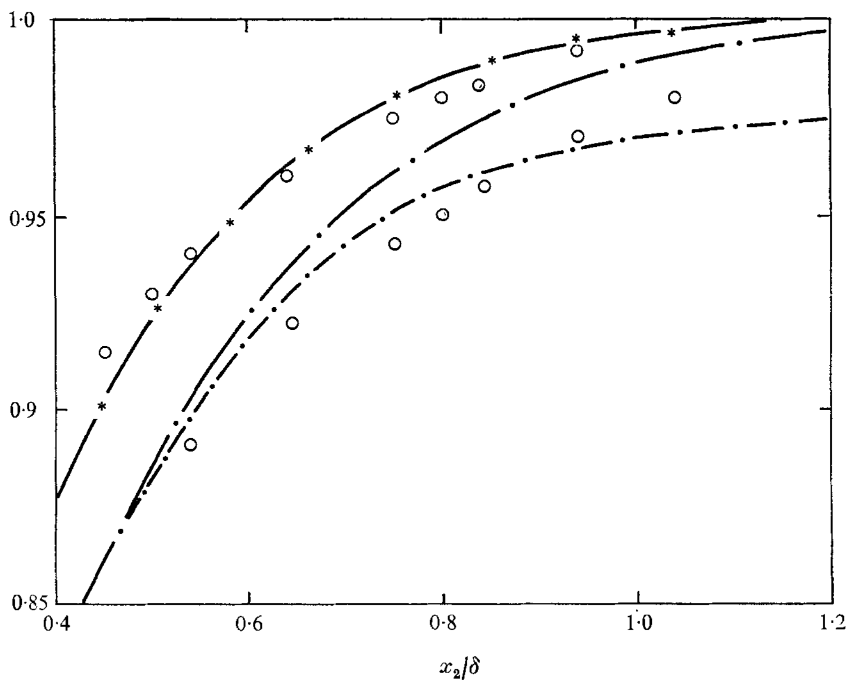

FIGURE 9. Distribution of zone averages of streamwise velocity components in a simulated boundary layer: —·—, $\bar{u}_1/U$; —·—·—, $\tilde{u}_1/\bar{I}$; —*—, $(\bar{u}_1 - \tilde{u}_1)(1-\bar{I})$; ○, experimental data from Kovasznay *et al.* (1970).

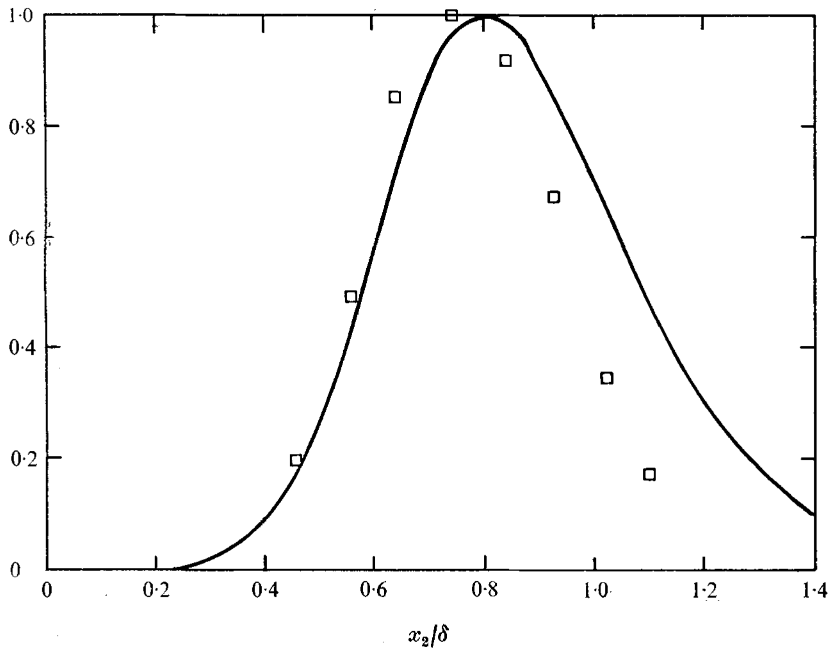

FIGURE 10. Distribution of normalized crossing frequency $f_I/(f_I)_{max}$ in a simulated boundary layer: ——, prediction; □, experimental data from Kovasznay *et al.* (1970).

$\overline{u}_2$ velocity components by computing the normal velocities in the turbulent fluid and in the external fluid relative to the unconditioned normal velocities. This corresponds to the presentation by Kovasznay *et al.* as necessitated by the small magnitude of the mean normal velocity component. In some sense, our results are in qualitative agreement with experiment. We find, for example, that the turbulent fluid is moving outward from the wall with respect to the unconditioned flow, and that therefore the external fluid is moving inward. However, the magnitudes of these differences are a decade smaller than the values shown by Kovasznay *et al.* In addition, of course, as we have discussed earlier, our analysis results in these differences vanishing as the intermittency vanishes.

In figure 10 we compare the predicted distribution of the crossing frequency, normalized with respect to this maximum value, with the corresponding data of Kovasznay *et al.* In this case, the distribution in the inner portions of the boundary layer is reasonably well represented, whereas in the outer portions the predicted crossing frequency is considerably higher than measured.

Finally, to obtain some insight into their relative importance, the distributions of the various modelled terms have been considered. It is found that, as suggested earlier, the creation term is dominant, the others being considerably smaller. This indicates that improvements in the agreement between prediction and experiment should be first sought in terms of improved models for the creation term.

It will be recalled that one of the advantages of testing similarity solutions is that values for the various length scales appearing in the modelling are obtained as part of the solutions. It is then appropriate to consider the values given in table 1 in this manner. We show in table 1 the 'edge' of the high-speed side of the mixing layer, and the 'edge' of the simulated boundary layer, in terms of η_e, defined as the value of η for which $f' = 0{\cdot}99$. The thickness of the layer can be estimated as $\delta = \eta_e x_1$. If the various lengths Λ_i introduced into the modelling are fixed multiples of δ_1, then $\beta\eta_e$, $\delta\eta_e$, $\kappa\eta_e$, $\phi\eta_e$ should be the same for each of our cases. These multiples can, of course, depend on other parameters of the flow. In fact, using the results of an additional free-mixing case not presented in detail, we find, within a factor of three and generally well within the range implied thereby, that

$$\beta\eta_e|T(0)| \simeq -0{\cdot}05, \quad \delta\eta_e(|T(0)|)^{\frac{1}{2}} \simeq -0{\cdot}04, \quad \kappa\eta_e(|T(0)|)^{\frac{3}{2}} \simeq 0{\cdot}2,$$
$$\phi\eta_e(|T(0)|)^{\frac{1}{2}} \simeq 1.$$

These estimates can be used as first approximations in calculations of other similar and non-similar flows.

With respect to the signs of the various parameters, it is encouraging that they are all consistent, in the sense that $\kappa > 0$, $\beta < 0$, $\delta < 0$ and $\phi > 0$ for all $\eta > 0$. Relative to the sign of δ it is interesting to note that this term arises from the modelling of the point statistic $\overline{u_k' \partial I/\partial x_k}$. From Antonia (1972) we conclude that, for $\eta > 0$, the greatest contribution to this term probably arises from $\overline{u_2' \partial I/\partial x_2}$, since $\overline{u_1' \partial I/\partial x_1}$ appears to be small, u_1' being the same at the upstream and downstream crossings. But, because of the large-scale turbulent structures, u_2' tends to be negative at downstream crossings for which $\partial I/\partial x_2 \rightarrow \infty$, and to be positive at upstream crossings for which $\partial I/\partial x_2 \rightarrow -\infty$. Thus, we expect

$$\overline{u_k' \partial I/\partial x_k} < 0$$

for $\eta > 0$. The opposite would be expected for $\eta < 0$. Clearly our results support these expectations. It does not appear possible to make similar remarks for the terms involving β and ϕ.

5. Concluding remarks

We have carried out an analysis of intermittent turbulent flow by developing a model equation which describes the turbulent fluid. Under the assumption that the unconditioned flow is known, the analysis permits the properties of the conditioned mean velocities (i.e. the mean values either within the turbulent fluid alone or within the external, irrotational fluid) and of the intermittency to be predicted. We compared predicted and experimental quantities for two mixing layers and for a simulated boundary layer. In many respects the agreement is quite satisfactory, but the need for further study of the several modelled terms, in particular that describing the creation of new turbulent fluid, is suggested by the predicted slower decay of the intermittency as the external flow is approached.

In addition to further refinements and developments of this analysis as it relates to the prediction of the conditioned flow behaviour when the unconditioned flow is assumed known (in particular for additional similar and for non-similar flows), it would appear highly desirable to consider the simultaneous treatment of the unconditioned and conditioned flows. This might be most worthwhile in connexion with the phenomenology of scalars.

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Note added in proof. After this manuscript had been accepted, Professor M. Morkovin questioned whether conditioning the continuity equation (1) would lead to an inconsistency among the resulting equations (6) and (11). The ensuing discussion resulted in the following illuminating considerations. If (1) is multiplied by I and the product averaged, we find simply

$$\overline{I\frac{\partial u_k}{\partial x_k}} = \frac{\partial}{\partial x_k}(\bar{u}_k)_1 - \overline{u_k\frac{\partial I}{\partial x_k}} = \bar{w} - \overline{u_n\frac{\partial I}{\partial x_n}},$$

where u_n is the instantaneous velocity component normal to the interface. We thus see that, at a fixed point in space, the mean rate of production of turbulent fluid is directly related to the statistical difference in the normal velocity component at upstream and downstream interface crossings. If u_n is decomposed into the velocity of the interface and the velocity relative to the interface and if the former contribution is statistically the same at upstream and downstream crossings, the equantity $\overline{u_n \partial I/\partial n}$ is equal to the average entrainment rate at both types of crossings. This strongly supports and clarifies our earlier ideas relative to $\bar{w}$ and furthermore suggests that the point of view that (6) is a model equation can well be replaced by the more positive view that (6) is an equation for the conservation of turbulent fluid. The author is grateful to Professor Morkovin for his stimulating question in particular and for his interest in this work in general.

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